39

AERODYNAMIC STABILITY DERIVATIVES OF AXISYMMETRIC BODY MOVING AT HYPERSONIC SPEEDS

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ABSTRACT

A method of calculating aerodynamic stability derivatives of bodies of revolution at high supersonic Mach numbers is presented. The method is an extension of linear perturbation method developed by M. Holt[1,2] and is applicable to the range of hypersonic parameter around unity where most of the existing supersonic and hypersonic theories lose their validity. Total flow field is composed of the known steady basic flow and the perturbation due to the motion of the body. Neglecting higher-order terms of the perturbation quantities a system of linear characteristic equations for the perturbation flow is derived. Solution is obtained by integration of these equations over the known network of characteristic curves for the basic flow. Time derivatives of flow variables contained in the equations are omitted by proper modifications of the boundary conditions thus replacing unsteady problem to steady one. Numerical calculations were made for case of steady pitching of a cone with semivertex angle of 10° in the range of hypersonic parameter near one. The results are compared to those by other existing theories and seem to indicate the utility of the present method in this range.

INTRODUCTION

In this paper is treated nonviscous unsteady flow due to slow motion of an axisymmetric body exposed in high supersonic flow. It is generally recognized that there is a great deal of difficulty in solving aerodynamic problems of a body moving at high supersonic Mach numbers. It is mainly because of this reason that linearization of the basic flow equations is no more valid in this region. The nature of high supersonic flow is determined primarily by the magnitude of hypersonic parameter, k, defined as product of the flow Mach number and the thickness ratio of the body considered. In case of k being far smaller than unity nature of the flow is truly supersonic and the problem is safely treated by the ordinary linearized supersonic theory. In extremely high Mach number range where k approaches infinity the underlying assumptions of the Newtonian impact theory become increasingly valid and reasonable solution to the problem can be obtained by comparatively simple calculation. Hypersonic problem in a real sense occurs when k is nearly equal to or greater than unity and difficulty in analytical treatment mostly lies in this region.

In hypersonic range even problems of nonviscous steady flow cannot so easily be treated and no complete method applicable to general cases has been established yet except for the characteristic method in which troublesome numerical calculations are unavoidably involved. To overcome the difficulty several approximate methods based on more or less semiempirical assumptions have been proposed. One is the generalized shock-expansion method by Eggers and Savin[3] and the other is the piston theory originated by Hayes.[4] These methods seem to be applicable to the calculation of stability derivatives in the range of k much higher than one.[4,5]

In these situations there seems to exist vacancy of proper analytical method based on the flow equations in the treatment of unsteady flow between the supersonic theory and the impact theory in the range where k is near unity. In this paper is developed a method of calculating stability derivatives of axisymmetric bodies with the intention of filling up the vacancy mentioned above. The method used here is an extension of the linear perturbation method developed by M. Holt.[1,2]

THE BASIC EQUATIONS

The total perturbation flow under consideration is composed of the basic flow or the steady flow around the axisymmetric body with zero angle of attack and the unsteady additive flow due to the unsteady motion of the body. It is assumed that perturbation variables are so small compared to the basic ones that their second- and higher-order terms are safely neglected.

Nondimensionalization of the physical quantities involved in the problem is made throughout the paper by using the following reference quantities:

Velocity: the limiting velocity, Q_l
Pressure: stagnation pressure in front of the shock wave, P_0
Density: stagnation density in front of the shock wave, R_0
Specific entropy: the gas constant, R
Length: body length, l
Time: l/Q_l

Let the cylindrical coordinates referred to the body axis with the origin at the vertex of the body be denoted by x, r and ψ. It is assumed that the motion of the body is restricted in the plane parallel to the meridian plane $\psi = 0$. Due to the axial symmetry of the basic flow and also due to the assumption of small perturbation, we can eliminate ψ from the basic equations and the boundary conditions by the use of the following notations and expressions for the flow variables in the basic and perturbation flows:

	Basic Flow	*Perturbation*
Total velocity	Q	$q \cos \psi$
x-component of velocity	U	$u \cos \psi$
r-component of velocity	V	$v \cos \psi$
ψ-component of velocity	W	$w \sin \psi$
Pressure	P	$p \cos \psi$
Density	R	$\rho \cos \psi$
Velocity of sound	C	$c \cos \psi$
Entropy	S	$s \cos \psi$

Thus we can treat the problem in the meridian plane $\psi = 0$.

Holt derived a system of characteristic equations for the perturbation flow and proved that the network of the characteristics for it is the same as that for the basic flow provided the higher-order terms are neglected. Hence the perturbation flow field is obtained in principle by integrating the linear characteristic equations over the known characteristic network for the basic flow.

Since the details of derivation of the system of linear equations are fully given in Holt's papers,[1,2] here we show directly the final results. The expression of equations given below is somewhat different from that by Holt. Characteristic equations:

$$\begin{aligned} - Q \sin \theta \frac{\partial u}{h_\beta \partial \beta} + Q \cos \theta \frac{\partial v}{h_\beta \partial \beta} - c\frac{w}{r} - c\frac{v}{r} - \frac{\gamma - 1}{2\gamma}\frac{\cot \mu}{R}\frac{\partial p}{h_\beta \partial \beta} \\ - \sin (\theta - \mu)\, \dot{u} + \cos (\theta - \mu)\dot{v} - \frac{\gamma - 1}{2\gamma}\frac{1}{RC}\dot{p} + K + L = 0 \end{aligned} \tag{1}$$

$$\begin{aligned} Q \sin \theta \frac{\partial u}{h_\alpha \partial \alpha} - Q \cos \theta \frac{\partial v}{h_\alpha \partial \alpha} - c\frac{w}{r} - c\frac{v}{r} - \frac{\gamma - 1}{2\gamma}\frac{\cot \mu}{R}\frac{\partial p}{h_\alpha \partial \alpha} \\ + \sin (\theta + \mu)\, \dot{u} - \cos (\theta + \mu)\, \dot{v} - \frac{\gamma - 1}{2\gamma}\frac{1}{RC}\dot{p} + K + M = 0 \end{aligned} \tag{2}$$

where

$$\begin{aligned} \frac{2\gamma}{\gamma - 1} K = - \frac{\operatorname{cosec} 2\mu}{RC}\Bigg[\Big\{u \sin (\theta + \mu) - v \cos (\theta + \mu)\Big\}\frac{\partial P}{h_\beta \partial \beta} \\ + \Big\{ - u \sin (\theta - \mu) + v \cos (\theta - \mu)\Big\}\frac{\partial P}{h_\alpha \partial \alpha}\Bigg] \\ + \frac{c}{RQ \sin^2 \mu \cos \mu}\left(\frac{\partial P}{h_\beta \partial \beta} + \frac{\partial P}{h_\alpha \partial \alpha}\right) \end{aligned} \tag{3}$$

$$L = \frac{\gamma - 1}{2\gamma}\frac{\rho}{R^2}\cot\mu\frac{\partial P}{h_\beta \partial\beta} + \operatorname{cosec} 2\mu \left\{ \sin(\theta+\mu) - v\cos(\theta+\mu)\right\}$$

$$\left\{-\sin(\theta-\mu)\frac{\partial U}{h_\beta\partial\beta} + \cos(\theta-\mu)\frac{\partial V}{h_\beta\partial\beta}\right\} + \operatorname{cosec} 2\mu\left\{-u\sin\right. \tag{4}$$

$$\left.(\theta-\mu) + v\cos(\theta-\mu)\right\}\left\{-\sin(\theta-\mu)\frac{\partial U}{h_\alpha\partial\alpha} + \cos(\theta-\mu)\frac{\partial V}{h_\alpha\partial\alpha}\right\}$$

$$M = \frac{\gamma - 1}{2\gamma}\frac{\rho}{R^2}\cot\mu\frac{\partial P}{h_\alpha\partial\alpha} + \operatorname{cosec} 2\,\mu\left\{u\sin(\theta+\mu) - v\cos(\theta+\mu)\right\}$$

$$\left\{\sin(\theta+\mu)\frac{\partial U}{h_\beta\partial\beta} - \cos(\theta+\mu)\frac{\partial V}{h_\beta\partial\beta}\right\} + \operatorname{cosec} 2\,\mu\left\{-u\sin(\theta-\mu)\right. \tag{5}$$

$$\left. + v\cos(\theta-\mu)\right\}\left\{\sin(\theta+\mu)\frac{\partial U}{h_\alpha\partial\alpha} - \cos(\theta+\mu)\frac{\partial V}{h_\alpha\partial\alpha}\right\}$$

In these equations α and β are the characteristic coordinates for the basic flow and the slopes of curves of β = constant (α-characteristic) and α = constant (β-characteristic) are equal to tan $(\theta + \mu)$ and tan $(\theta - \mu)$, respectively, where θ and μ are the flow direction angle and the Mach angle, respectively, in the basic flow. $h_\alpha d\alpha$ and $h_\beta d\beta$ are length of the line elements along α- and β-characteristics, respectively. γ is the ratio of specific heats.

Relation between entropy and vorticity:

$$\frac{\partial w}{\partial\lambda} = -\frac{Vw}{rQ} + \frac{\gamma-1}{\gamma}\frac{p}{2QRr} - \frac{\dot w}{Q} \tag{6}$$

where $d\lambda$ is the projection of streamline element to the meridian plane $\psi = 0$, and dot denotes time derivative.

Entropy relation:

$$\dot s = Q\frac{\partial s}{\partial\lambda} + q\frac{\partial S}{\partial\lambda} \tag{7}$$

Equation of state:

$$s + \frac{1}{\gamma-1}\left(\frac{p}{P} - \gamma\frac{\rho}{R}\right) = 0 \tag{8}$$

Energy equation:

$$uU + vV + \frac{\gamma-1}{2\gamma}\int\frac{dp}{R} - \frac{\gamma-1}{2\gamma}\int\frac{\rho}{R^2}\,dP = \frac{\gamma-1}{2\gamma}\int\frac{\dot p}{RQ}\,d\lambda \tag{9}$$

Eqs. (1), (2) and (6) to (9) form a system of simultaneous equations for six unknown variables, u, v, w, p, ρ and s.

BOUNDARY CONDITIONS

The boundary conditions of the problem are given on the shock wave and on the body surface. Since we deal with flow around a cone, only shock wave issuing from the vertex of the body is considered, and hence velocity components in front of the shock wave are $U = Q_0$, $V = W = 0$ where Q_0 is the free-stream velocity. In an unsteady problem the shock wave itself must naturally be unsteady and change its shape and position. However, in the present treatment, approximation is made that the shock wave is steady. Since slightly modified body shape is used in the boundary condition as will be explained later, the shape of the shock wave somewhat deforms from the original conical one. The velocity relation across a steady shock wave is given as

$$\left.\begin{aligned} \frac{\cos\Omega}{\cos(\Omega-\theta)} &= \frac{Q}{Q_0} \\ \frac{1}{\tan\theta} &= \left[\frac{\gamma+1}{2}\frac{M_0^2}{M_0^2\sin\Omega-1}-1\right]\tan\Omega \end{aligned}\right\} \tag{10}$$

where M_0 is the free-stream Mach number and is related to Q_0 as

$$\frac{1}{Q_0^2} = 1 + \frac{2}{\gamma-1}\frac{1}{M_0^2} \tag{11}$$

and Ω is the shock wave angle. Using Eq. (10) perturbation velocities, u and v, behind the shock wave are obtained as functions of change of shock-wave angle, $\Delta\Omega$. w is expressed in terms of $\Delta\Omega$ by the following approximation:

$$w = Q_0\,\Delta\Omega \tag{12}$$

p is also determined by $\Delta\Omega$. In the actual calculation shock condition works as an alternate of a β-characteristic.

We have the condition on the body surface stating that there is no relative motion normal to the body. Here, we consider the following three kinds of motion of a cone with semivertex angle of $\tan^{-1}\tau$: (1) sinking with uniform vertical velocity, (2) pitching with uniform angular velocity around the nose, and (3) sinking with uniform vertical acceleration. For each case approximate body shape and the boundary condition are given as follows:

(1) Sinking with uniform vertical velocity (equivalent to stationary angle of attack case).

$$\text{body shape:}\qquad r = \tau x + U\alpha \tag{13}$$

$$\text{boundary condition:}\qquad \tau u - v + U\alpha = 0 \tag{14}$$

where α is the corresponding angle of attack.

(2) Pitching with uniform angular velocity.

$$\text{body shape:}\qquad r = \tau x + qx^2/2Q_0 \tag{15}$$

$$\text{boundary condition:}\qquad \tau u - v + Uqx/Q_0 = 0 \tag{16}$$

where q is the pitching rate.

(3) Sinking with uniform vertical acceleration.

$$\text{body shape:} \qquad r = \tau x + \dot{\alpha}\, tx \tag{17}$$

$$\text{boundary condition:} \qquad \tau u - v + \dot{\alpha} x + U\, \dot{\alpha} t = 0 \tag{18}$$

where ($\dot{\alpha}$) is time derivative of the corresponding angle of attack, and t is time.

In cases of (1) and (2) the boundary conditions given above do not contain time, and hence, the problems can be treated as steady ones for slightly deformed bodies from the original cone. On the other hand, in case (3), time enters in the boundary condition and the problem is essentially unsteady. However, when t approaches zero, terms containing t in Eqs. (17) and (18) are very small compared to other terms. Neglecting these terms the boundary conditions for case (3) are replaced by the following steady equations:

$$\text{body shape:} \qquad r = \tau x \tag{19}$$

$$\text{boundary condition:} \qquad \tau u - v + \dot{\alpha} x = 0 \tag{20}$$

Therefore, even in case (3) the problem can be taken as steady provided t is very small. The stability derivatives thus obtained give us their limiting values at $t \to 0$. In this way we can get solutions for the three cases mentioned above by treating steady flow around modified bodies under proper boundary conditions, thus avoiding difficulty to be met in dealing with the time derivative terms in the basic equations.

CALCULATION PROCEDURE AND NUMERICAL EXAMPLES

The procedure of calculating perturbation flow field by the present method is mostly the same as that used in ordinary characteristic method. In the first place the basic flow is obtained by the characteristic method and all coefficients in the system of the basic equations [Eqs. (1) to (9)] are evaluated at each cross-point of the characteristic curves. Next, w and s are calculated by integrating Eqs. (6) and (7) along streamline. ρ is expressed in terms of s and p in Eq. (8). w, s and ρ thus obtained are substituted in Eqs. (1), (2) and (9), and we get three simultaneous equations for u, v and p. The process required at the shock wave and on the body surface is not different from that used in the ordinary charac-

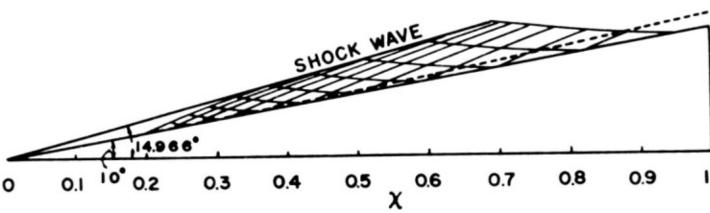

Fig. 1. Characteristic network for a cone of 10° semivertex angle at $M_0 = 0.5453$.

teristic method. Since the basic equations are linear in perturbation variables the above mentioned calculation over a known characteristic network is generally much easier then the process required to determine the basic flow field.

An application was made to steady pitching case for a cone with semivertex angle of 10 deg. The free-stream Mach numbers chosen are 4.0951, 5.4526, and 6.745. The corresponding values of hypersonic parameter are 0.72, 0.95, and 1.08, respectively. Assuming that $q/2Q_0$ is equal to 0.017453, Eqs. (15) and (16) are written as

$$\left.\begin{aligned} r &= 0.17633x + 0.017453x^2 \\ 0.17633u - v + 0.034907Ux &= 0 \end{aligned}\right\} \qquad (21)$$

Hence, near the nose where x is very close to zero the original body shape is retained and the perturbation quantities are nearly equal to zero. Therefore, it is practical to start the step-by-step calculation at an α-characteristic curve issuing from the body surface at say $x = 0.2$ putting $u = v = w = p = \rho = s = 0$ on it.

The characteristic network for the cone at $M_0 = 5.4526$ is shown in Fig. 1. The dotted line in the figure indicates the modified body shape given by Eq. (21). Distributions of additive pressure p on the body are presented in Fig. 2. The

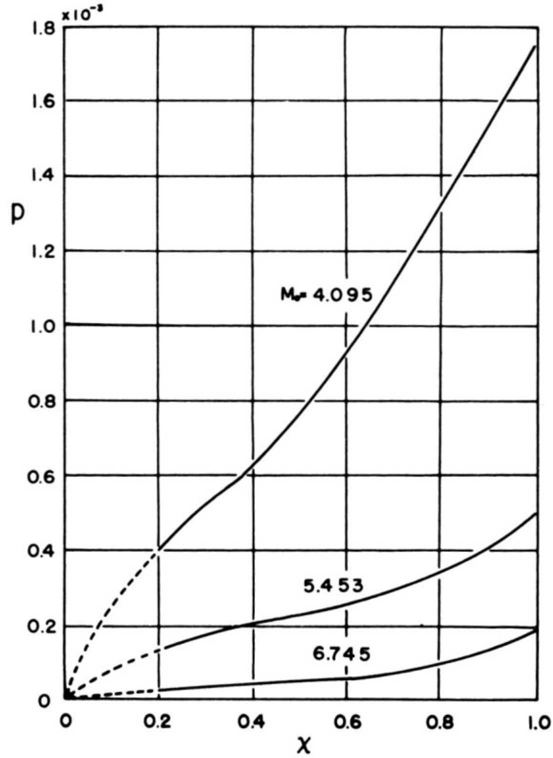

Fig. 2. Additive pressure distribution on the cone at $q/2Q = 0.01745$.

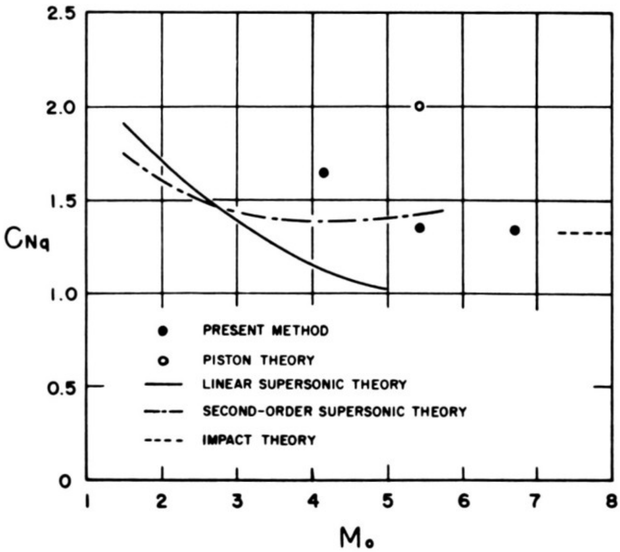

Fig. 3. Variation with Mach number of the stability derivative.

actual additive pressure on the body surface is given as $p \cos \psi$, and normal force is obtained by the integration of additive pressure over the entire surface of the cone. The stability derivative C_{Nq} due to the pitching motion is defined as

$$C_{Nq} = C_N/(q/Q_0)$$

where C_N is the normal force coefficient referred to the body base area. In Fig. 3 are plotted the results of the present calculation together with theoretical results of linear supersonic theory,[6] second-order supersonic theory,[7] and Newtonian impact theory. One value of C_{Nq} calculated by the piston theory is also shown in the figure.

The second-order supersonic theory generally shows divergence before the limit of its applicability is reached, and there seems to exist a gap between the second-order supersonic theory and the impact theory in the range of hypersonic parameter around unity. The present numerical results seem to fill up the gap thus indicating the utility of the present method in this region.

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Discussor: Milton Van Dyke, Stanford University

The authors are to be commended for having attacked the problem of the oscillating cone, which has needed solving for some time. It may be observed that impact theory, which is actually only an empirical estimate, does not invariably yield accurate results for very high Mach number. Thus MacIntosh at Stanford has recently solved the complementary problem of the oscillating wedge (in the hypersonic small-disturbance approximation) and finds the Newtonian estimate of stability derivatives to be considerably in error.